

LEADER ELECTION: A MARKOV CHAIN APPROACH

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ABSTRACT. A well-studied randomized election algorithm proceeds as follows: In each round the remaining candidates each toss a coin and leave the competition if they obtain heads. Of interest is the number of rounds required and the number of winners, both related to maxima of geometric random samples, as well as the number of remaining participants as a function of the number of rounds. We introduce two related Markov chains and use ideas and methods from discrete potential theory to analyse the respective asymptotic behaviour as the initial number of participants grows. One of the tools used is the approach via the Rényi-Sukhatme representation of exponential order statistics, which was first used in the leader election context by Bruss and Grübel in [BG03].

1. INTRODUCTION

We consider the following election algorithm: Starting with a group of size n , in each round the remaining participants simultaneously toss a coin and leave the competition if it turns up heads, which we suppose to happen with probability $\theta \in (0, 1)$ for the coins of all members of the group. In its simplest form the procedure ends if there is only one person left, in which case the winner is unique, or if all remaining participants obtain heads in the same round, when there would consequently be more than one winner.

This simple random election algorithm and its variants have received quite some attention over a period of more than a quarter of a century, see e.g. [BO90, ESS93, BES95, FMS96, KP96, BG03, Gne04, LP09, KM14, AKM15]. The duration of the original game is obviously related to the maximum of a sample of geometric random variables, and the probability that the winner is unique is similarly related to the uniqueness of this maximum. The earliest published paper on this topic that we are aware of is the paper [BO90] by Bruss and O’Cinneide, who refer to a presentation by P.N. Bajaj at an AMS meeting in 1988. They found that the probability that there is a single winner, i.e. that the maximum is unique, does not converge as $n \rightarrow \infty$ and that it is asymptotically logarithmically periodic, meaning in particular that the probability does converge along specific subsequences $(n_k)_{k \in \mathbb{N}}$.

As most of the authors cited above, we are interested in the behaviour of the election algorithm as the number n of participants grows to infinity, specifically in the use of Markov chain techniques. We now describe two chains that are relevant in this context.

First, we may think of an infinite number of participants who all toss their coins simultaneously, and we then regard the first n of these. Specifically, with a sequence $(\xi_i)_{i \in \mathbb{N}}$ of independent random variables, all geometrically distributed with parameter θ , and with

$$M_n := \max\{\xi_1, \dots, \xi_n\},$$

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we obtain a representation of the number R_n of rounds needed and the number L_n of winners as

$$(1) \quad L_n := \#\{1 \leq i \leq n : \xi_i = M_n\}, \quad R_n = \begin{cases} M_n, & \text{if } L_n = 1, \\ M_n + 1, & \text{if } L_n \geq 2, \end{cases}$$

if we start with n participants (non-uniqueness of the maximum is noticed only after an additional round, with the number of participants dropping from some $k > 1$ to 0). The point here is that the values for different n 's are 'coupled' in a manner that leads to a Markov chain $(Y_n)_{n \in \mathbb{N}}$, $Y_n := (M_n, L_n)$, with state space $E = \mathbb{N} \times \mathbb{N}$ and transition probabilities

$$(2) \quad \mathbb{P}(Y_{n+1} = (j, l) | Y_n = (i, k)) = \begin{cases} \theta(1-\theta)^{j-1}, & \text{if } j > i, l = 1, \\ \theta(1-\theta)^{i-1}, & \text{if } j = i, l = k+1, \\ 1 - (1-\theta)^{i-1}, & \text{if } j = i, l = k, \\ 0, & \text{otherwise.} \end{cases}$$

Secondly, we consider the number N_n of participants in round n if we start with a group of size k , so that $N_1 = k$. The process $(N_n)_{n \in \mathbb{N}}$ is again a Markov chain, now with state space $\{0, \dots, n\}$ and transition probabilities

$$(3) \quad \mathbb{P}(N_{n+1} = j | N_n = i) = p(i, j) := \begin{cases} \binom{i}{j} \theta^{i-j} (1-\theta)^j, & \text{if } i > 0, j = 0, \dots, i, \\ 1, & \text{if } i = j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $N_n \rightarrow 0$ with probability 1 as $n \rightarrow \infty$, whatever the initial number k , so asymptotics will refer to a sequence of such processes with $k \rightarrow \infty$ in this case.

We will use discrete potential theory for the analysis of the space-time behaviour of these chains. This area is also known as Markov chain boundary theory and has recently found many applications in the context of random combinatorial structures that arise in the analysis of sequential algorithms; see [Grü13] for a simple introduction. Our results give further examples for the use of this approach. The chains in the present paper are more complicated than those of the combinatorial type as we no longer have a locally finite transition tree, meaning that from a given state there are infinitely many possible next states that can be visited with positive probability.

The next section contains the necessary boundary theory background. In Sections 3 and Sections 4 respectively we then apply the theory to the two chains introduced above. In terms of the election algorithm the first application relates to the existence of strong limit results for the duration and the number of winners, the second contributes, in our view, to the understanding of the periodicity phenomenon mentioned above. We will skip some technical details and refer the reader to [Hag16] for a full treatment, together with various extensions.

2. BOUNDARY THEORY FOR SPACE-TIME MARKOV CHAINS

Doob's seminal paper [Doo59] may be regarded as the starting point of boundary theory for Markov processes with discrete time parameter and discrete state space. A recent and excellent textbook introduction to this circle of ideas is contained in [Woe09], but see also the classic [KSK76]. We give a brief outline of the main ideas, but omit details.

The basic data consist of

- a countable set E , the state space,

- a probability measure on E , the initial distribution, represented by a sequence $q = (q(x))_{x \in E}$ where $q(x)$ is the probability of the set $\{x\}$, and
- a matrix $P = (p(x, y))_{x, y \in E}$ of functions $p : E \times E \rightarrow \mathbb{R}_+$, the transition matrix and transition probabilities, where the latter satisfy

$$\sum_{y \in E} p(x, y) = 1 \quad \text{for all } x \in E.$$

Let $\Omega := E^{\mathbb{N}}$ be the set of all sequences $(x_n)_{n \in \mathbb{N}}$ of elements of E . We endow Ω with the σ -field \mathcal{A} generated by the sets

$$A(x_1, \dots, x_k) := \{(y_n)_{n \in \mathbb{N}} \in \Omega : y_i = x_i \text{ for } i = 1, \dots, k\}$$

where $k \in \mathbb{N}$ and $x_1, \dots, x_k \in E$, and refer to the measurable space (Ω, \mathcal{A}) as the path space. The basic data listed above provide a unique probability measure \mathbb{P} on the path space via

$$(4) \quad \mathbb{P}(A(x_1, x_2, \dots, x_k)) = q(x_1) \prod_{i=2}^k p(x_{i-1}, x_i),$$

for all $k \in \mathbb{N}$, $x_1, \dots, x_k \in E$. Using the projections

$$X_n : \Omega \rightarrow E, (x_m)_{m \in \mathbb{N}} \mapsto x_n,$$

$n \in \mathbb{N}$, we then obtain a Markov chain $(X_n)_{n \in \mathbb{N}}$ with state space E , initial distribution q and transition matrix P . We assume a weak form of irreducibility,

$$(5) \quad \mathbb{P}(X_n = x \text{ for some } n \in \mathbb{N}) > 0 \quad \text{for all } x \in E,$$

which means that every state has a positive probability of being visited.

In this set-up we say that the chain is of *space-time type* if the time parameter n of the chain is a function of its state $x \in E$. Equivalently we may consider the state space as being graded in the sense that E is the disjoint union of the segments E_n of possible values of X_n . In particular, the transition mechanism is then adapted to the grading in the sense that $p(x, y) > 0$ implies that $x \in E_n, y \in E_{n+1}$ for some $n \in \mathbb{N}$. It is well-known (and trivial) that any Markov chain $(Y_n)_{n \in \mathbb{N}}$, if augmented by the time parameter, gives a space-time chain $(X_n)_{n \in \mathbb{N}}$, i.e. we set $X_n := (n, Y_n)$ for all $n \in \mathbb{N}$. Further, for a Markov chain $(Y_n)_{n \in \mathbb{N}}$ that is not homogeneous in time, meaning that the transition probabilities may depend on n , this augmentation provides time homogeneity.

The above framework gives rise to three structures, with the corresponding problems of describing these in terms of familiar objects.

For the first we recall that a function $h : E \rightarrow \mathbb{R}$ is *harmonic* if

$$h(x) = \sum_{y \in E} p(x, y) h(y) \quad \text{for all } x \in E.$$

We are interested in the set $\mathbb{H}_{1,+}$ of such functions that are non-negative and normalized in the sense that $\sum_{x \in E} q(x) h(x) = 1$. This is a convex set, so describing $\mathbb{H}_{1,+}$ could be its identification as a Choquet simplex together with a characterization of its extreme points.

For the second object and problem we define the backwards transition matrix P^{co} and transition probabilities $p^{\text{co}}(y, x)$, $x, y \in E$, by

$$p^{\text{co}}(y, x) := \mathbb{P}(X_{n-1} = x | X_n = y), \quad P^{\text{co}} = (p^{\text{co}}(y, x))_{y, x \in E}.$$

These are also known as *cotransition probabilities*. The object of interest is now the set \mathcal{M} of all probability measures \mathbb{P} on the path space under which the projections $(X_n)_{n \in \mathbb{N}}$ become a Markov chain with the same backwards transitions as under the original \mathbb{P} . A straightforward computation shows that \mathcal{M} is again a convex set.

Finally, by a *compactification* of the state space we mean a compact topological space \bar{E} together with an injective mapping $\phi : E \rightarrow \bar{E}$ such that the image $\phi(E)$ is dense in \bar{E} and that the trace of the \bar{E} topology on this image is equal to the discrete topology. The third structure we are interested in is a compactification of the state space in which the X_n 's converge almost surely as $n \rightarrow \infty$, and that is sufficiently detailed in the sense that the limit variable X_∞ generates the tail σ -field of the process, meaning that

$$\sigma(X_\infty) =_{\text{a.s.}} \mathcal{T}(X) := \bigcap_{n=1}^{\infty} \sigma(\{X_m : m \geq n\}).$$

Interestingly, for space-time Markov chains these three questions are closely related. First, for an $h \in \mathbb{H}_{1,+}$ we may define a new transition mechanism, leading to the h -transform of the original chain, by

$$(6) \quad P_h = (p_h(x, y))_{x, y \in E}, \quad p_h(x, y) = \frac{1}{h(x)} p(x, y) h(y).$$

In order to not lose irreducibility in the sense of (5), we have to restrict the state space from E to the set $E(h) := \{x \in E : h(x) > 0\}$. Let \mathbb{P}_h be the associated measure on the path space, so that under \mathbb{P}_h the process $(X_n)_{n \in \mathbb{N}}$ is a Markov chain with transition matrix P_h . It is easy to see that (6) extends to n -step transitions, which leads to

$$(7) \quad \frac{d\mathbb{P}_h^{(X_1, \dots, X_k)}}{d\mathbb{P}^{(X_1, \dots, X_k)}}(x) = h(x_k) \quad \text{for all } x = (x_n)_{n \in \mathbb{N}} \in \Omega.$$

Another straightforward calculation shows that the transformed process has the same backwards transition probabilities as \mathbb{P} , so that $\mathbb{P}_h \in \mathcal{M}$. Conversely, for any $\tilde{\mathbb{P}} \in \mathcal{M}$ we obtain an element $h \in \mathbb{H}_{1,+}$ via

$$(8) \quad h(x) := \frac{\tilde{\mathbb{P}}(X_n = x)}{\mathbb{P}(X_n = x)} \quad \text{for all } x \in E_n.$$

Note that (5) is important here, and that the definition of h relies on the gradedness of the state space. With the help of (7) it is easy to check that, apart from being bijective, the relationship is also linear in the sense that

$$\mathbb{P}_{\alpha h_1 + (1-\alpha)h_2} = \alpha \mathbb{P}_{h_1} + (1-\alpha) \mathbb{P}_{h_2} \quad \text{for all } h_1, h_2 \in \mathbb{H}_{1,+}, 0 < \alpha < 1.$$

As a consequence the h -transform maps the extreme points of the convex sets $\mathbb{H}_{1,+}$ and \mathcal{M} to each other.

The starting point for the connection between harmonic functions and the third problem is the observation that $(h(X_n), \mathcal{F}_n)_{n \in \mathbb{N}}$ is a non-negative martingale for all $h \in \mathbb{H}_{1,+}$; here \mathcal{F}_n is the σ -field generated by the variables X_1, \dots, X_n . Thus, by Doob's forward convergence theorem, $h(X_n)$ converges to some limit variable Z almost surely, where of course Z may depend on h . Conversely, for an event $A \in \mathcal{T}$ with $\kappa := \mathbb{P}(A) > 0$ we obtain an element h of $\mathbb{H}_{1,+}$ via the following steps: By the Markov property, we may write the conditional expectation of the corresponding indicator function as a function of X_n , $\phi_n(X_n) = E[1_A | \mathcal{F}_n]$, and the space-time property implies that we may then consistently define h on the whole of E , as in (8), by setting $h(x) = \kappa^{-1} \phi_n(x)$ for $x \in E_n$, $n \in \mathbb{N}$.

The specific *Doob-Martin compactification* now proceeds as follows: We regard a sequence $(y_n)_{n \in \mathbb{N}} \subset E$ as convergent if, for all fixed $m \in \mathbb{N}$ and $x_1, \dots, x_m \in E$, the conditional probabilities $\mathbb{P}(X_1 = x_1, \dots, X_m = x_m | X_n = y_n)$ converge as $n \rightarrow \infty$. Due to the Markov

property the construction can be based on the *Martin kernel* K ,

$$K(x, y) := \frac{\mathbb{P}(X_n = y | X_m = x)}{\mathbb{P}(X_n = y)}, \quad x, y \in E, n > m,$$

where m and n are the time values associated with the states x and y respectively. Indeed,

$$\mathbb{P}(X_1 = x_1, \dots, X_m = x_m | X_n = y_n) = K(x_m, y_n) \mathbb{P}(X_1 = x_1, \dots, X_m = x_m),$$

which connects the convergence condition for the conditional probabilities to the convergence of the values of the Martin kernel, and which also exhibits the connection to the backwards transitions. The functions $K(x, \cdot)$, $x \in E$, are bounded and separate the points of E . The Stone-Ćech procedure from general topology, or the introduction of a suitable metric on E and subsequent completion, then lead to a compact space \bar{E} that contains (a copy of) the discrete space E and allows for a continuous extension of the functions $y \mapsto K(x, y)$, $x \in E$. Using the same symbol K for the extended functions and lower case Greek letters for the boundary elements we then obtain that all extremal elements $\mathbb{H}_{1,+}$ are of the form $K(\cdot, \alpha)$ for some α in the *Martin boundary* $\partial E := \bar{E} \setminus E$. In many cases the reverse implication also holds; in general one writes $\partial_{\min} E$ for the subset corresponding to the extremal elements; this is the *minimal* boundary. With this notation in place, we can now state some central results:

- (R1) For each $h \in \mathbb{H}_{1,+}$ there exists a unique probability measure μ_h on (the Borel subsets of) \bar{E} with $\mu_h(\partial_{\min} E) = 1$ that represents h in the sense of

$$h(x) = \int K(x, \alpha) \mu_h(d\alpha) \quad \text{for all } x \in E.$$

- (R2) With respect to \mathbb{P}_h the variables X_n converge almost surely to some X_∞ with values in the boundary.
 (R3) The limit variable X_∞ generates the tail σ -field up to \mathbb{P}_h -null sets.
 (R4) The distribution of X_∞ is given by the measure μ_h that represents h .
 (R5) The process conditioned on $X_\infty = \alpha \in \partial_{\min} E$ is an h -transform of the original chain, with the harmonic function given by $h = K(\cdot, \alpha)$.

The last of these may be rephrased as follows: The kernel

$$Q : \partial_{\min} E \times \mathcal{A} \rightarrow [0, 1], \quad (\alpha, A) \mapsto \mathbb{P}_{K(\cdot, \alpha)}(A)$$

is a regular version of the conditional distribution of X given X_∞ . This leads to an interpretation of the chain as a two-stage experiment, where we first select the final value and then run the corresponding transformed chain. There is an obvious analogy with classical and Bayesian statistics; see [Dyn78] and [Lau88].

We end this sketch with a disclaimer: The beauty and elegance of the general theory notwithstanding, its actual implementation for a specific Markov chain, such as a description of the boundary and the conditioned chains, may be far from being trivial; see [EGW16] for a recent example. Independent of their applications in the context of the election algorithm it may therefore be of interest that the two chains introduced in Section 1 permit a comparably short and explicit treatment along the above lines.

3. THE MAXIMUM AND ITS MULTIPLICITIES

Recall the definition (1) of our first Markov chain, with transition probabilities as given in (2). We write $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ for the one-point compactification of \mathbb{N} .

Theorem 1. *Let $X = (X_n)_{n \in \mathbb{N}}$, with $X_n := (n, M_n, L_n)$ for all $n \in \mathbb{N}$, be the space-time chain associated with the process of maxima and their multiplicities.*

(a) A sequence $(x_n)_{n \in \mathbb{N}}$ of states $x_n = (n, j_n, l_n) \in E$ converges in the Doob-Martin topology associated with X if and only if, as $n \rightarrow \infty$,

- j_n converges to some $J \in \bar{\mathbb{N}}$, and
- l_n/n converges to some $\alpha \in [0, 1]$ in the euclidean topology.

(b) The extended Martin kernel $K(\cdot; J, \alpha) : E \rightarrow \mathbb{R}_+$ associated with the limit point $(J, \alpha) \in \bar{\mathbb{N}} \times [0, 1]$ is given by

$$\begin{aligned} K(m, i, k; J, \alpha) &= (1 - \alpha)^m (1 - (1 - \theta)^{J-1})^{-m}, \quad \text{if } J \in \mathbb{N}, J > i, \\ K(m, J, k; J, \alpha) &= \alpha^k (1 - \alpha)^{m-k} (\theta(1 - \theta)^{J-1})^{-k} (1 - (1 - \theta)^{J-1})^{k-m}, \quad \text{if } J \in \mathbb{N}, \\ K(m, i, k; \infty, \alpha) &= (1 - \alpha)^m, \end{aligned}$$

and $K(m, i, k; J, \alpha) = 0$ in all other cases.

(c) As $n \rightarrow \infty$, X_n converges almost surely to the fixed boundary point $(\infty, 0)$.

(d) The extended Martin kernels associated with (J, α) are harmonic if $J \in \mathbb{N}$, and the corresponding h -transform is the Markov chain with transition probabilities

$$(9) \quad p^{(h)}(m, i, k; m+1, j, l) = \begin{cases} \frac{(1 - \alpha)(1 - (1 - \theta)^{i-1})}{1 - (1 - \theta)^{J-1}}, & \text{if } i = j \leq J, l = k, \\ \frac{(1 - \alpha)(1 - \theta)^{i-1} \theta}{1 - (1 - \theta)^{J-1}}, & \text{if } j = i < J, l = k + 1, \\ \frac{(1 - \alpha)\theta(1 - \theta)^{j-1}}{1 - (1 - \theta)^{J-1}}, & \text{if } i < j < J, l = 1, \\ \alpha, & \text{if } i < j, j = J, l = 1, \\ \alpha, & \text{if } i = j = J, l = k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The Markov chain in Part (d) arises by conditioning on the limits J for the maximum and α for its relative multiplicity. A constructive interpretation can be obtained as follows: First, consider a sequence $(\xi_n^J)_{n \in \mathbb{N}}$ of independent random variables, which all have the geometric distribution with parameter θ , but now conditioned on the range $\{1, 2, \dots, J-1\}$. Let $Y^J = (Y_n^J)_{n \in \mathbb{N}}$ be the bivariate process of maxima and their multiplicities based on $(\xi_n^J)_{n \in \mathbb{N}}$. Consider a second chain $Z^J = (Z_n^J)_{n \in \mathbb{N}}$ that moves from (i, k) , $i < J$, to $(J, 1)$ with probability 1, and then moves upwards on the vertical line $\{J\} \times \mathbb{N}$ with probability α resp. stays where it is with probability $1 - \alpha$. As long as we are inside the strip $\{1, \dots, J-1\} \times \mathbb{N}$ we follow Y^J with probability $1 - \alpha$ but switch to Z^J with probability α . Note the double role of the parameter α .

The extended Martin kernel associated with a pair (∞, α) is not harmonic but only superharmonic, which means that the corresponding transition matrix is strictly substochastic. Nevertheless we can arrive at an interpretation along the lines just given for (J, α) with $J \in \mathbb{N}$ if we augment the state space by admitting the value ∞ for the maximum: The transformed process would be based on the same ξ -variables as the original process, but we would jump to $(\infty, 1)$ with probability α and then stay or move upwards on $\{\infty\} \times \mathbb{N}$ as in the case $J \in \mathbb{N}$.

For the proof of the theorem we need the following elementary lemma.

Lemma 2. *Let $m \in \mathbb{N}$ be fixed and consider a sequence $(l_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ with $l_n \leq n - m$ for all $n \in \mathbb{N}$. Then $\binom{n-m}{l_n} \binom{n}{l_n}^{-1}$ converges if and only if l_n/n converges, both as $n \rightarrow \infty$. Further,*

if $\lim_{n \rightarrow \infty} l_n/n = \alpha$, then

$$(10) \quad \lim_{n \rightarrow \infty} \binom{n-m}{l_n} \binom{n}{l_n}^{-1} = (1-\alpha)^m,$$

and, for all $k \in \mathbb{N}$,

$$(11) \quad \lim_{n \rightarrow \infty} \binom{n-m}{l_n-k} \binom{n}{l_n}^{-1} = \alpha^k (1-\alpha)^m.$$

Proof. We have

$$(12) \quad \binom{n-m}{l_n} \binom{n}{l_n}^{-1} = \prod_{r=0}^{l_n-1} \frac{n-m-r}{n-r} = \exp\left(\sum_{r=0}^{l_n-1} \log\left(1 - \frac{m}{n-r}\right)\right).$$

Suppose that $l_n/n \rightarrow \alpha \in (0, 1)$ and let

$$R(n, m, r) := \log\left(1 - \frac{m}{n-r}\right) + \frac{m}{n-r}.$$

We will use the inequality $|\log(1-x) + x| \leq x^2$ which is valid for $0 \leq x \leq 1/2$. In view of $\alpha < 1$ we can find $n_0 = n_0(m)$ such that $0 \leq m/(n-r) \leq 1/2$ for all $n \geq n_0$, and for such n

$$\sum_{r=0}^{l_n-1} |R(n, m, r)| \leq \frac{m^2 l_n}{(n-l_n)^2},$$

which is $o(1)$ in view of $\alpha < 1$. Further,

$$\sum_{r=0}^{l_n-1} \frac{1}{n-r} = \int_0^{l_n/n} \frac{1}{1 - \frac{[nx]}{n}} dx \rightarrow \int_0^\alpha \frac{1}{1-x} dx = -\log(1-\alpha),$$

so that, taken together,

$$\lim_{n \rightarrow \infty} \sum_{r=0}^{l_n-1} \log\left(1 - \frac{m}{n-r}\right) = -m \log(1-\alpha).$$

Hence the limit of the ratio of the binomial coefficients exists and is equal to $(1-\alpha)^m$ if $l_n/n \rightarrow \alpha \in (0, 1)$. Monotonicity consideration can be used to extend this to the boundaries $\alpha = 0$ and $\alpha = 1$. Taken together this proves (10); the modifications needed for (11) should be obvious.

For the proof of the convergence condition we note that

$$\alpha_- := \liminf_{n \rightarrow \infty} \frac{l_n}{n} < \limsup_{n \rightarrow \infty} \frac{l_n}{n} =: \alpha_+$$

would imply the existence of two subsequence such that the corresponding ratios of binomial coefficients in (10) converge to $(1-\alpha_-)^m$ and $(1-\alpha_+)^m$ respectively. \square

Proof of Theorem 1. In order to have maximal value j with multiplicity l at time n we need exactly l of the variables ξ_1, \dots, ξ_n to be equal to j and all others to be strictly less than j . This gives

$$\mathbb{P}(M_n = j, L_n = l) = \binom{n}{l} (\theta(1-\theta)^{j-1})^l (1 - (1-\theta)^{j-1})^{n-l}.$$

Similarly, an advance from $(M_m, L_m) = (i, k)$ to $(M_n, L_n) = (i, l)$ with $l \geq k$ has probability

$$\mathbb{P}(M_n = i, L_n = l | M_m = i, L_m = k)$$

$$= \binom{n-m}{l-k} (\theta(1-\theta)^{i-1})^{l-k} (1 - (1-\theta)^{i-1})^{n-m-(l-k)},$$

so that

$$(13) \quad K(m, i, k; n, i, l) = \binom{n-m}{l-k} \binom{n}{l}^{-1} (\theta(1-\theta)^{i-1})^{-k} (1 - (1-\theta)^{i-1})^{k-m}.$$

If $l = l_n$ depends on n such that $l_n/n \rightarrow \alpha \in [0, 1]$ then, by Lemma 2, for any fixed m, i, k this converges as $n \rightarrow \infty$ to

$$K(m, i, k; \infty, i, \alpha) = \alpha^k (1-\alpha)^{m-k} (\theta(1-\theta)^{i-1})^{-k} (1 - (1-\theta)^{i-1})^{k-m}.$$

Similarly, if the value of the maximum increases, so that $j > i$, then for $l = 1, \dots, n-m$,

$$\begin{aligned} \mathbb{P}(M_n = j, L_n = l | M_m = i, L_m = k) \\ = \binom{n-m}{l} (\theta(1-\theta)^{j-1})^l (1 - (1-\theta)^{j-1})^{n-m-l}, \end{aligned}$$

and we arrive at

$$(14) \quad K(m, i, k; n, j, l) = \binom{n-m}{l} \binom{n}{l}^{-1} (1 - (1-\theta)^{j-1})^{-m}.$$

By Lemma 2 again, if $l_n/n \rightarrow \alpha \in [0, 1]$, then this converges for any fixed m, i, k if $j > i$ and the limit is

$$K(m, i, k; \infty, j, \alpha) = (1-\alpha)^m (1 - (1-\theta)^{j-1})^{-m}.$$

In the other direction this lemma, together with (13) and (14), also shows that in order for $K(m, i, k; n, J, l_n)$ to converge for $i = 1, \dots, J$ it is necessary that l_n/n converges to some value in $[0, 1]$.

Now suppose that $(j_n)_{n \in \mathbb{N}}$ is such that $j_n \rightarrow \infty$ as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} (1 - (1-\theta)^{j_n})^{-m} = 1 \quad \text{for all } m \in \mathbb{N}.$$

This together with (14) shows that $K(m, i, k; n, j_n, l_n)$ converges for all m, i, k if and only if $\binom{n-m}{l_n} \binom{n}{l_n}^{-1}$ converges, which implies the $J = \infty$ part of the first assertion.

For (a) and (b) it remains to show that for sequences $(j_n)_{n \in \mathbb{N}}$ that do not converge in the one-point compactification of \mathbb{N} convergence of $K(m, i, k; n, j_n, l_n)$ cannot hold for all m, i, k , regardless of the behaviour of $(l_n)_{n \in \mathbb{N}}$.

If $(j_n)_{n \in \mathbb{N}}$ does not converge in $\bar{\mathbb{N}}$ then there are sequences $(n_s)_{s \in \mathbb{N}}, (n_t)_{t \in \mathbb{N}} \subset \mathbb{N}$ with $n_s, n_t \rightarrow \infty$ such that

$$(15) \quad J_- := \lim_{s \rightarrow \infty} j_{n_s} = \liminf_{n \rightarrow \infty} j_n < \limsup_{n \rightarrow \infty} j_n = \lim_{t \rightarrow \infty} j_{n_t} =: J_+.$$

In particular, $J_- \in \mathbb{N}$ and then necessarily, for some $s_0 \in \mathbb{N}$, $j_{n_s} = J_-$ for all $s \geq s_0$. Convergence of (n, j_n, l_n) in the Doob-Martin topology means that $K(m, i, k; n, j_n, l_n)$ converges for all m, i and k . For $m = 1$ we obtain

$$(16) \quad K(1, i, k; n, j, l) = \frac{1 - l/n}{1 - (1-\theta)^j} \quad \text{if } i < j,$$

and $K(1, i, k; n, j, l) = 0$ if $i > j$. In view of (15) both alternatives happen infinitely often for the sequence $(j_n)_{n \in \mathbb{N}}$ if we choose i such that $J_- < i < J_+$, so that only the value 0 is

possible for the limit. Since the denominator in (16) is bounded by 1 and also bounded away from 0, we must therefore have $\lim_{n \rightarrow \infty} l_n/n = 1$. However, this in turn implies

$$\lim_{s \rightarrow \infty} K(1, J_-, 1; n_s, j_{n_s}, l_{n_s}) = \lim_{s \rightarrow \infty} K(1, J_-, 1; n_s, J_-, l_{n_s}) = \theta^{-1} (1 - \theta)^{J_- - 1} > 0,$$

which is a contradiction.

The remaining case where $J_+ = J_- + 1$ can be handled similarly.

The above proves parts (a) and (b). For the proof of (c) we first note that the general theory implies that $X_n = (M_n, L_n) \rightarrow X_\infty$ almost surely for some X_∞ with values in the boundary. For M_n , it is therefore enough to prove convergence in distribution, which is straightforward from $\mathbb{P}(M_n > k) \rightarrow 1$ for all fixed $k \in \mathbb{N}$. Clearly, $L_n = 1$ for infinitely many n on the set $M_n \uparrow \infty$, so the almost sure limit for L_n/n can only have the value 0.

Finally, (d) follows by calculation. \square

Boundary theory for randomly growing discrete structures can sometimes be used to amplify a known result on distributional convergence to an almost sure statement; see e.g. [Grü14] in connection with the Wiener index of search trees. Obviously such a strengthening to a pathwise result is not possible if the tail σ -field of the sequence of interest is trivial in the sense that it consists of events with probability 0 or 1 only. The above result therefore shows that the known distributional limit theorems for the maximum and its multiplicity (along suitable subsequences) do not hold with probability one. Indeed, there are no deterministic transformations of M_n or L_n , even if we allow additional dependence on n or pass to a subsequence, that lead to a non-degenerate strong limit.

Remark 3. (a) Boundary theory can lead to an explicit description of the tail σ -field of a sequence of random variables, if these constitute a Markov chain. However, if interest is mainly in the qualitative aspect of tail triviality then there are general results that can be used. Indeed, in the case at hand it is easy to see that the terminal σ -field associated with the sequence $(M_n, L_n)_{n \in \mathbb{N}}$ is contained in the exchangeable σ -field associated with the sequence $(\xi_i)_{i \in \mathbb{N}}$, and, by the Hewitt-Savage theorem, see e.g. [Bre68, Section 3.9], i.i.d. sequences have a trivial exchangeable σ -field.

(b) It is known, see e.g. [Woe09, Corollary 7.51], that a boundary point is an element of $\partial_{\min} E$ if and only if the tail σ -field of the corresponding h -transform is trivial. The explicit construction for $h = K(\cdot, \cdot; \infty, J, \alpha)$ given above makes it possible to use the Hewitt-Savage theorem to show that all these boundary points are in fact minimal. \triangleleft

4. THE NUMBER OF PARTICIPANTS

The election algorithm motivates the following question: If we start at time $n = 1$ with a group of size j , what is the number of remaining participants, considered as a function of the number of rounds carried out? For example, the total duration of the procedure may then be written as the entrance time of the process into the set $\{0, 1\}$, see (1).

Again, we are interested in asymptotics, meaning here that $j = j_k$ is large. This situation differs from the one in Section 3 where we considered the asymptotic behaviour of a fixed chain. Two possibilities offer themselves: We can ‘invert time’ and aim for a description of the set \mathcal{M} defined in Section 2, or we shift the first time index from $n = 1$ to some negative value depending on j , with the hope that a limit object emerges for the resulting sequence of processes. We will do both.

For the first approach we start with the state space $E := \mathbb{N} \times \mathbb{N}_0$ and consider Markov chains X with backwards transition probabilities

$$(17) \quad \mathbb{P}(X_n = (n, j) | X_{n+1} = (n+1, i)) = \binom{i}{j} (1-\theta)^j \theta^{i-j}$$

for all $n \in \mathbb{N}$, $i \in \mathbb{N}_0$ and $j \in \{0, \dots, i\}$. To simplify the notation we introduce

$$c(\theta) := -\frac{1}{\log(1-\theta)}.$$

Note that $c(\theta) \log y$ is equal to the logarithm of $y > 0$ with respect to the base $1/(1-\theta)$. We recall that the associated state space augmentation is a compact topological space that is unique only up to homeomorphisms. In the representation given below we write \mathbb{R}^* for the one-point (!) compactification of the real line. Formally, we have some object \diamond that is not an element of \mathbb{R} , we set $\mathbb{R}^* = \mathbb{R} \sqcup \{\diamond\}$, we retain the euclidean topology on \mathbb{R} , and, finally, we regard any sequence $(x_n)_{n \in \mathbb{N}}$ with

$$\#\{n \in \mathbb{N} : x_n \in [a, b]\} < \infty \quad \text{for all } a, b \in \mathbb{R}$$

as convergent with limit \diamond . We will also need the probability densities f_l , $l \in \mathbb{N}$, given by

$$(18) \quad f_l(x) = \frac{1}{(l-1)!} \exp(-lx - e^{-x}), \quad x \in \mathbb{R},$$

and Euler's constant $\gamma := \lim_{n \rightarrow \infty} (H_n - \log n)$, where $H_n := \sum_{k=1}^n 1/k$ denotes the n th harmonic number.

Theorem 4. *Let X be a Markov chain with backwards transition probabilities given by (17).*

(a) *The Doob-Martin boundary ∂E of E associated with the space-time version of X is \mathbb{R}^* , where a sequence of states (n_k, j_k) , $k \in \mathbb{N}$, converges to $z \in \mathbb{R}^*$ if and only if*

$$(19) \quad \lim_{k \rightarrow \infty} (c(\theta) \log(j_k) - n_k) = z.$$

(b) *Let W_i and ζ_i be independent random variables, where ζ_i is exponentially distributed with parameter i and W_i has density f_i , see (18). Then, for $z \in \mathbb{R}$, the extended Martin kernel can be written as*

$$(20) \quad K(m, i; z) = \frac{1}{\mathbb{P}(X_m = i)} \mathbb{P}(W_i < c_\infty(m, i; z) < W_i + \zeta_i)$$

for all $m \in \mathbb{N}$, $i \in \mathbb{N}$, and with

$$(21) \quad c_\infty(m, i; z) := H_i - \gamma - \frac{m+z}{c(\theta)}.$$

Further, $K(m, i; \diamond) \equiv 0$.

(c) *The extended Martin kernels $K(\cdot, \cdot; z)$ are harmonic if $z \in \mathbb{R}$.*

Proof. We have

$$K(m, i; n, j) = \frac{\mathbb{P}(X_n = j | X_m = i)}{\mathbb{P}(X_n = j)} = \frac{\mathbb{P}(X_m = i | X_n = j)}{\mathbb{P}(X_m = i)},$$

so we need the n -step transition probabilities associated with

$$p(i, j) = \begin{cases} \binom{i}{j} \theta^{i-j} (1-\theta)^j, & i > 0, j = 0, \dots, i, \\ 1, & i = j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We consider first the case $j_k \rightarrow \infty$. Again, let $\xi_l, l \in \mathbb{N}$, be independent random variables, all geometrically distributed with parameter θ . Then, with

$$(22) \quad \kappa(m, i; n_k, j_k) := \mathbb{P}(X_m = i | X_{n_k} = j_k) = \mathbb{P}(X_m = i) K(m, i; n_k, j_k)$$

we have $\kappa(m, i; n_k, j_k) = \mathbb{P}(\#\{1 \leq l \leq j_k : \xi_l > n_k - m\} = i)$, for all $j_k \geq i, n_k \geq m$. As in [BG03] we now use the well-known relation between geometric and exponential random variables together with the Rényi-Sukhatme representation of the order statistics of a sample from an exponential distribution; see e.g. [SW86, p.336] for the latter. For this, we start with a sequence $(\eta_i)_{i \in \mathbb{N}}$ of independent random variables, all exponentially distributed with parameter 1. Then $(\lceil c(\theta)\eta_i \rceil)_{i \in \mathbb{N}}$ is equal in distribution to $(\xi_i)_{i \in \mathbb{N}}$, and

$$\begin{aligned} \#\{1 \leq l \leq j_k : \xi_l \geq n_k - m\} &= \#\{1 \leq l \leq j_k : \lceil c(\theta)\eta_l \rceil > n_k - m\} \\ &= \#\{1 \leq l \leq j_k : \eta_l \geq c(\theta)^{-1}(n_k - m)\}, \end{aligned}$$

Writing $\eta_{(j_k:1)} < \eta_{(j_k:2)} < \dots < \eta_{(j_k:j_k)}$ for the increasing order statistics associated with $\eta_1, \dots, \eta_{j_k}$ we therefore have that

$$(23) \quad \kappa(m, i; n_k, j_k) = \mathbb{P}(\eta_{(j_k:j_k-i)} \leq c(\theta)^{-1}(n_k - m) < \eta_{(j_k:j_k-i+1)}).$$

The Rényi-Sukhatme representation says that the random vector $(\eta_{(j_k:1)}, \eta_{(j_k:2)}, \dots, \eta_{(j_k:j_k)})$ is equal in distribution to the random vector

$$(V_{j_k}, V_{j_k} + V_{j_k-1}, \dots, V_{j_k} + V_{j_k-1} + \dots + V_1),$$

with V_1, \dots, V_{j_k} independent, and where V_l is exponentially distributed with parameter l , $l = 1, \dots, j_k$. In particular,

$$\eta_{(j_k:j_k-i)} =_{\text{distr}} V_{i+1} + V_{i+2} + \dots + V_{j_k}.$$

It is easy to see that $(M_{j_k,i})_{j_k > i}$ with

$$M_{j_k,i} := \sum_{l=i+1}^{j_k} \left(V_l - \frac{1}{l} \right), \quad j_k > i,$$

is a martingale that is bounded in L^2 . Hence $M_{j_k,i}$ converges almost surely as $k \rightarrow \infty$ to some square integrable random variable W_i . The representation further implies that $\eta_{(j_k:j_k-i+1)} - \eta_{(j_k:j_k-i)}$ is independent of $\eta_{(j_k:j_k-i)}$ and that it has an exponential distribution with parameter i . Taken together this gives the following convergence in distribution,

$$\left(\eta_{(j_k:j_k-i)} - \sum_{l=i+1}^{j_k} \frac{1}{l}, \eta_{(j_k:j_k-i+1)} - \sum_{l=i+1}^{j_k} \frac{1}{l} \right) \rightarrow_{\text{distr}} (W_i, W_i + \zeta_i) \quad \text{as } k \rightarrow \infty,$$

with W_i and ζ_i independent, and ζ_i exponentially distributed with parameter i . This implies the convergence of the probabilities in (23) if

$$c_k(m, i) := \frac{n_k - m}{c(\theta)} - \sum_{l=i+1}^{j_k} \frac{1}{l}$$

converges (a detailed argument would use Slutsky's Lemma together with the continuity of the distribution functions of W_i and $W_i + \zeta_i$). Recalling the expansion $H_j = \log j + \gamma + o(1)$ as $j \rightarrow \infty$, we see that this is equivalent to the convergence of $c(\theta) \log j_k - n_k$ to some $z \in \mathbb{R}$, and that we then have

$$\lim_{k \rightarrow \infty} c_k(m, i) = c_\infty(m, i; z)$$

with $c_\infty(m, i; z)$ as in (21), and consequently

$$\lim_{k \rightarrow \infty} \kappa(m, i; n_k, j_k) = \mathbb{P}(W_i < c_\infty(m, i; z) < W_i + \eta_i).$$

The results in [BG03, p.1258] imply that W_i has density f_i . Taken together this shows that the convergence in (a) implies the convergence of the Martin kernels, together with the formula given in (20).

It remains to show that the convergence condition in (a) is also necessary. As in the proof of Theorem 1 we recall that Doob-Martin convergence implies the convergence of the Martin kernels for *all* m and i . Below we will use $i = 1$ and choose m large enough.

Suppose that we have

$$-\infty < z_- := \liminf_{k \rightarrow \infty} (c(\theta) \log(j_k) - n_k) < \limsup_{k \rightarrow \infty} (c(\theta) \log(j_k) - n_k) =: z_+ < +\infty.$$

In particular, with suitably chosen subsequences $(k_+(l))_{l \in \mathbb{N}}$ and $(k_-(l))_{l \in \mathbb{N}}$ we would obtain z_+ and z_- as limits of $c(\theta) \log(j_k) - n_k$ along $k = k_+(l)$ and $k = k_-(l)$ as $l \rightarrow \infty$. From $z_- < z_+$ it follows that $c_\infty(m, 1; z_+) < c_\infty(m, 1; z_-)$ for all $m \in \mathbb{N}$. We now note that the density of W_1 is strictly positive and unimodal, with $\arg\max$ at 0. This implies that the function

$$y \mapsto \mathbb{P}(W_1 < y < W_1 + \zeta_1) = \int_0^\infty (\mathbb{P}(W_1 \leq y) - \mathbb{P}(W_1 \leq y - s)) e^{-s} ds$$

is strictly increasing on an interval $(-\infty, 0)$. With m chosen large enough, both $c_\infty(m, 1; z_+)$ and $c_\infty(m, 1; z_-)$ are less than 0, so that

$$\mathbb{P}(W_1 < c_\infty(m, 1; z_+) < W_1 + \eta_1) < \mathbb{P}(W_1 < c_\infty(m, 1; z_-) < W_1 + \eta_1).$$

Putting things together we see that $K(m, 1; n_{k_+(l)}, j_{k_+(l)})$ and $K(m, 1; n_{k_-(l)}, j_{k_-(l)})$ converge to different values. Hence (n_k, j_k) does not converge in the Doob-Martin topology.

Similar arguments work in the other cases, where z_- and z_+ may not be finite.

Further, the argument in the necessity proof can also be used to show that different z -values lead to different extended kernels, and it is straightforward to show that pointwise convergence of a sequence $(K(\cdot, z_n))_{n \in \mathbb{N}}$ of extended kernels is equivalent to the convergence of $(z_n)_{n \in \mathbb{N}}$ in \mathbb{R}^* .

Part (c) follows by calculation, see [Hag16]. \square

In contrast to the situation in Section 3 we did not start with a specific chain X or probability measure \mathbb{P} on (Ω, \mathcal{A}) , so it is not clear whether a chain with the postulated cotransitions exists at all.

Example 5. Let us write $\text{Geo}_0(\eta)$ for the number of failures version of the geometric distribution with parameter, so that $V \sim \text{Geo}_0(\eta)$ means

$$\mathbb{P}(V = i) = (1 - \eta)^i \eta \quad \text{for all } i \in \mathbb{N}_0.$$

Let W be another random variable, defined on the same probability space as V , with

$$\mathbb{P}(W = j | V = i) = \binom{i}{j} (1 - \theta)^j \theta^{i-j} \quad \text{for all } i \in \mathbb{N}_0, j \in \{0, \dots, i\}.$$

Taken together, the distribution of V and the conditional distribution of W given V determine the distribution of W . After some straightforward calculations we obtain

$$\mathbb{P}(W = j) = \sum_{i=j}^{\infty} \binom{i}{j} (1 - \theta)^j \theta^{i-j} (1 - \eta)^i \eta = \left(1 - \frac{\eta}{1 - \theta + \eta \theta}\right)^j \frac{\eta}{1 - \theta + \eta \theta},$$

which shows that $W \sim \text{Geo}_0(\zeta)$ with $\zeta := \eta / (1 - \theta + \eta \theta)$.

The inverse ψ_θ of the function $\eta \mapsto \zeta$ is given by

$$\psi_\theta(\zeta) = \frac{\zeta(1-\theta)}{1-\zeta\theta}.$$

It is easy to check that ψ_θ is continuous and strictly increasing, with $\psi_\theta(0) = 0$, $\psi_\theta(1) = 1$. For a given $\zeta_1 \in (0, 1)$ we define the sequence $(\zeta_n)_{n \in \mathbb{N}}$ recursively by $\zeta_{n+1} = \psi_\theta(\zeta_n)$ for all $n \in \mathbb{N}$. The above calculation, together with Kolmogorov's consistency theorem, now provides the existence of a probability measure \mathbb{P}_ζ on the path space such that, under this measure, $(X_n)_{n \in \mathbb{N}}$ is a Markov chain with the required backwards transitions; moreover, by construction $X_n \sim \text{Geo}_0(\zeta_n)$ for all $n \in \mathbb{N}$. This chain is not homogeneous in time, but our basic object is the corresponding space-time chain $((n, X_n))_{n \in \mathbb{N}}$, which is automatically homogeneous in time; see [Hag16] for more details. \triangleleft

For the second approach we consider a sequence of Markov chains with transition probabilities as in (3) and with start in state j_k at time $-k+1$; here we regard the sequence $(j_k)_{k \in \mathbb{N}}$ as given. For a formal treatment we replace the time range \mathbb{N} that we have used so far by the full set \mathbb{Z} of all integers. The path space is now $\Omega = \mathbb{N}_0^{\mathbb{Z}}$, the projections give a two-sided sequence $(X_n)_{n \in \mathbb{Z}}$, these generate the σ -field \mathcal{A} on Ω , and probability measures on (Ω, \mathcal{A}) are determined by the values they assign to sets of the form

$$(24) \quad A(i_l, \dots, i_m) = \{(j_n)_{n \in \mathbb{Z}} \in \Omega : i_s = j_s \text{ for } s = l, \dots, m\},$$

with $l, m \in \mathbb{Z}$, $l < m$, and $i_l, \dots, i_m \in \mathbb{N}_0$. In particular, for each $k \in \mathbb{N}$ we can define a probability measure \mathbb{P}_k on (Ω, \mathcal{A}) via

$$\mathbb{P}_k(A(i_l, \dots, i_m)) = \prod_{s=-k+1}^{m-1} p(i_s, i_{s+1}),$$

with p as in (3), whenever $l \leq -k+1$ and $i_l = \dots = i_{-k+1} = j_k$; to sets with $i_l \neq j_k$ for some $l \leq -k+1$ we assign the value 0. Under \mathbb{P}_k , the coordinate process $(X_n)_{n \in \mathbb{Z}}$ models the sequence of participant numbers if we have a fixed number j_k up to time $-k+1$ and then start the coin-tossing selection procedure. Clearly, the paths are decreasing and the state 0 is absorbing. Because of the extension to time values before $-k+1$ these measures are all defined on the same measurable space. If we endow \mathbb{N}_0 with the discrete topology and Ω with the corresponding product topology then we obtain a topological space so that weak convergence of probability measures is defined. It is easy to see that a sequence of probability measures on (Ω, \mathcal{A}) converges weakly with respect to this topology if and only if the probabilities of all sets of the type given in (24) converge.

The following result is essentially a reformulation of parts of Theorem 4 from this alternative point of view.

Theorem 6. *For a given sequence $(j_k)_{k \in \mathbb{N}}$ of positive integers let $(\mathbb{P}_k)_{k \in \mathbb{N}}$ be the associated sequence of probability measures on $\mathbb{N}_0^{\mathbb{Z}}$, where \mathbb{P}_k is such that $\mathbb{P}_k(X_n = j_k) = 1$ for all $n \leq -k+1$, and $(X_n)_{n \geq -k+1}$ is a Markov chain with transition probabilities as in (3).*

Then $(\mathbb{P}_k)_{k \in \mathbb{N}}$ converges weakly as $k \rightarrow \infty$ if and only if, for some $z \in \mathbb{R}$,

$$\lim_{k \rightarrow \infty} (c(\theta) \log j_k - k) = z.$$

Under the limit measure \mathbb{P}^z the full sequence $(X_n)_{n \in \mathbb{Z}}$ is a Markov chain with transitions as in (3), and, almost surely with respect to \mathbb{P}^z ,

$$\lim_{n \rightarrow -\infty} (c(\theta) \log X_n + n) = z.$$

Proof. Let $A = A(i_l, \dots, i_m)$ be as in (24). Then, for all $k \in \mathbb{N}$ with $-k < l$, and with κ as in (22),

$$\mathbb{P}_k(A) = \kappa(-l, i_l; k, j_k).$$

In particular, weak convergence as $k \rightarrow \infty$ of \mathbb{P}_k is equivalent to convergence of $\kappa(-l, i_l; k, j_k)$ for all $l \in -\mathbb{N}$, $i_l \in \mathbb{N}_0$, so that the first statement is an immediate consequence of part (a) of Theorem 4.

Clearly, the Markov property is not lost and neither does the transition mechanism change when passing to the limit \mathbb{P}^z of a converging sequence $(\mathbb{P}_k)_{k \in \mathbb{N}}$. This gives the second statement of the theorem. To obtain the third, we note that $(Y_n)_{n \in \mathbb{N}}$ with $Y_n := X_{-n}$ for all $n \in \mathbb{N}$ is a Markov chain with backwards transition probabilities as given in (17). The general convergence result (R2) mentioned at the end of Section 2 now implies that $Y_n \rightarrow z$ as $n \rightarrow \infty$, \mathbb{P}^z -almost surely and with respect to the Doob-Martin convergence. Hence another invocation of Theorem 4 completes the proof. \square

We return to the election algorithm: If we start with a group of size j_k then the number of rounds needed is the maximum M_{j_k} of a sample of size j_k from the geometric distribution with parameter θ if the maximum is unique, and $M_{j_k} + 1$ otherwise. It is well known that, if $j_k \rightarrow \infty$ with $k \rightarrow \infty$, we then need to subtract suitable values $n_k \in \mathbb{N}$, with $n_k \rightarrow \infty$ as $k \rightarrow \infty$, to obtain a tight sequence of distributions $\mathcal{L}(M_{j_k} - n_k)$, $k \in \mathbb{N}$, and that convergence in distribution only holds along specific subsequences. Indeed, in the limit a logarithmic periodicity can be observed; see Remark 3 for the non-existence of strong limit results for the maxima.

On first sight, the results of this section do not seem to contribute to our understanding of this periodicity phenomenon that many authors found intriguing and that may have contributed to the popularity of the election algorithm in the mathematical literature. In order to obtain a connection we introduce the shift operator on the new path space,

$$T : \mathbb{N}_0^{\mathbb{Z}} \rightarrow \mathbb{N}_0^{\mathbb{Z}}, \quad (i_n)_{n \in \mathbb{Z}} \mapsto (i_{n+1})_{n \in \mathbb{Z}}.$$

Here is the basic observation: Replacing k by $k + 1$ in (19) corresponds to replacing z by $z - 1$, which implies $(\mathbb{P}_z)^T = \mathbb{P}_{z+1}$. On the process side, the shift T corresponds to the transition from $(X_n)_{n \in \mathbb{Z}}$ to $(X_{n+1})_{n \in \mathbb{Z}}$. In particular, the probability of events such as the time from some initial number of participants to the first entry into $\{0, 1\}$ is invariant under T , which shows that the distribution of the duration of the algorithm depends on z only via its fractional part.

As in the previous section we close this section with some pointers to related results.

Remark 7. (a) A sample of size n from the geometric distribution with parameter θ can be obtained by putting balls into an infinite sequence of urns in the following way: In the first round, each of the initially n balls is put into the first urn with probability θ , then each of the remaining balls is put into the second urn with probability θ , and so on. Gnedin [Gne04] introduced a randomized version, the Bernoulli sieve, where the fixed values θ for the successive rounds are replaced by the values of a sequence of independent and identically distributed random variables. Interestingly, such an additional randomisation may eliminate the oscillatory effects inherent to the classical geometric leader election; see [GINR09] and [GIM10].

(b) In Section 2 we dealt with what is known as the *exit* boundary. The step from Theorem 4 to Theorem 6 corresponds to the somewhat dual notion of an *entrance* boundary; see [Doo59] and [KSK76].

(c) Markov chain boundaries have been thoroughly investigated in the context of random walks on discrete structures, a standard reference being [Woe00]. Branching (or

Galton-Watson) processes are another class of discrete time Markov chains where a lot is known about the boundaries; see [AN72, Chapter II]. This class is of special relevance to the above process of participant numbers, as this process is in fact a branching process with a specific offspring distribution: There is either one descendant or none at all, with respective probabilities $1 - \theta$ and θ . For such subcritical cases a general result on the entrance boundary has been obtained in [AR06]; in particular, for the process itself (not the space-time version) the boundary is the torus under quite general conditions. \triangleleft

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